

BURNING OF A POWDER UNDER HARMONICALLY VARYING PRESSURE

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A powder model is examined in which the surface temperature of the powder T_s is a function of the pressure and the initial temperature T_0 . All the processes taking place in the reactive layer of the condensed phase and in the gas phase are assumed to be inertialess. It is shown that the equation of heat conduction with a convection term, the boundary conditions for which relate the burning rate, the surface temperature, and the temperature gradient at the burning rate can be treated in the same terms as oscillations described by ordinary differential equations. It is meaningful to introduce such concepts as natural frequency, damping constant, and resonance.

1. Model of burning of powders. Zel'dovich's [1] theory of burning of powders includes a method for studying nonsteady combustion processes. The main point of the method is that the steady-state dependence of the burning rate $m^0(T_0, p)$ on the initial temperature T_0 and pressure p can be reduced to the relation $m(f, p)$, where f is the temperature gradient at the powder surface. The functional relationship obtained holds for nonsteady conditions also (the superscript on the burning rate is therefore neglected), because the gradient defines the temperature in the combustion zone on which the burning rate depends. The transition from $m^0(T_0, p)$ to $m(f, p)$ is accomplished with the aid of the known relation between the gradient, burning rate, and initial temperature

$$\kappa f^0 = \frac{m^0}{\rho} (T_s^0 - T_0), \tag{1.1}$$

which holds in steady-state conditions (κ is the thermal diffusivity and ρ is the powder density). It is natural that in such an approach to the study of non-steady phenomena the inertia properties of all processes with exception of heat conductivity in the condensed phase should be neglected.

In Zel'dovich's theory, the surface temperature of the powder T_s is taken as constant, which means that it is independent of both pressure and the initial temperature of the powder. The author [2] has shown that on the same assumption concerning the principal role of the inertia properties of the heated layer of the condensed phase, it is possible in an analogous manner to take into account the dependence of the surface temperature on T_0 and p in theory of nonsteady burning of powders. Thus, in [3] it was shown that in nonsteady conditions, T_s also depends on the gradient and pressure, and that this dependence $T_s(f, p)$ can be derived from the steady-state law $T_s^0(T_0, p)$ by taking (1.1) into account.

The present work deals, in the linear approximation, with the process of burning of powders at variable pressure. We will need for this purpose a

relation between the first derivatives of the burning rate with respect rate to T_0 and p , and to f and p :

$$k = (T_s^0 - T_0) \left(\frac{\partial \ln m^0}{\partial T_0} \right)_p, \quad r = \left(\frac{\partial T_s^0}{\partial T_0} \right)_p, \tag{1.2}$$

$$v = \left(\frac{\partial \ln m^0}{\partial \ln p} \right)_{T_0}, \quad \mu = \frac{1}{T_s^0 - T_0} \left(\frac{\partial T_s^0}{\partial \ln p} \right)_{T_0}.$$

The transition from these quantities, which can be determined from steady-state combustion experiments, to derivatives with respect to f and p can be achieved most easily with the aid of Jacobians. From relation (1.1) we have

$$(T_s^0 - T_0) \left(\frac{\partial \ln f^0}{\partial T_0} \right)_p = k + r - 1, \quad \left(\frac{\partial \ln f^0}{\partial \ln p} \right)_{T_0} = v + \mu.$$

Hence, for example

$$\begin{aligned} \left(\frac{\partial \ln m^0}{\partial \ln p} \right)_{f_0} &= \frac{\partial (\ln m^0, \ln f^0)}{\partial (\ln p, \ln f^0)} = \frac{\partial (\ln m^0, \ln f^0) / \partial (\ln p, T_0)}{\partial (\ln p, \ln f^0) / \partial (\ln p, T_0)} = \\ &= \frac{(\partial \ln m^0 / \partial \ln p)_{T_0} (\partial \ln f^0 / \partial T_0)_p - (\partial \ln m^0 / \partial T_0)_p (\partial \ln f^0 / \partial \ln p)_{T_0}}{(\partial \ln f^0 / \partial T_0)_p} = \\ &= \frac{v(r-1) - \mu k}{k+r-1}. \end{aligned}$$

In the same fashion we can obtain the remaining relations that are given below, and which in virtue of the above hold also in the nonsteady case

$$\begin{aligned} \left(\frac{\partial \ln m}{\partial \ln p} \right)_f &= \frac{v(r-1) - \mu k}{k+r-1}, \quad \frac{1}{T_s^0 - T_0} \left(\frac{\partial T_s}{\partial \ln p} \right)_f = \frac{\mu(k-1) - vr}{k+r-1}, \\ \left(\frac{\partial \ln m}{\partial \ln f} \right)_p &= \frac{k}{k+r-1}, \quad \frac{1}{T_s^0 - T_0} \left(\frac{\partial T_s}{\partial \ln f} \right)_p = \frac{r}{k+r-1}. \end{aligned} \tag{1.3}$$

For $r = \mu = 0$, the expression (1.3) reduces to Zeldovich's relations [3].

$$\left(\frac{\partial \ln m}{\partial \ln f} \right)_p = \frac{k}{k-1}, \quad \left(\frac{\partial \ln m}{\partial \ln p} \right)_f = \frac{v}{1-k}.$$

2. Eigenfrequency and attenuation constant. Upon investigation of the combustion stability of the model under examination, it was found [2] that the assumption concerning the dependence of T_s^0 on T_0 leads to a substantial extension of the region of stable combustion. Whereas for $r = 0$ (Zel'dovich's case), the steady-state regime is stable only if $k < 1$, for $r > 0$, stable combustion is also possible at $k > 1$, in which case the stability region is defined by the relation $r \geq (k-1)^2 / (k+1)$. In this context, it was found that the relaxation of nonsteady temperature distribution and burning rate to steady-state values occurs in an

oscillatory fashion (the presence of oscillatory modes of powder combustion was first noted in a paper by A. G. Istratov and V. B. Librovich [4]). At $k > 1$ and $r \geq (k-1)^2/(k+1)$, the time factor that defines the approach to the steady regime has the form $[\Omega(u^0)^2 t / \kappa]$, where u^0 is the steady-state value of the linear burning rate $u^0 = m^0/\rho$, and

$$\Omega = \frac{(k-1)^2 - r(k+1)}{2r^2} \pm i \frac{k-1}{2r^2} \sqrt{(2k-r+2)r - (k-1)^2}. \quad (2.1)$$

It is evident that $\lambda = -\text{Re } \Omega$ characterizes the attenuation of the oscillations and it is therefore natural to denote by

$$\lambda = \frac{r(k+1) - (k-1)^2}{2r^2} \quad (2.2)$$

the damping constant of the oscillations. The imaginary part of Ω , which defines the frequency of the oscillations, can be written in the form

$$\text{Im } \Omega = \pm \sqrt{\omega^2 - \lambda^2} \quad (\omega = \sqrt{k/r}). \quad (2.3)$$

We will use ω to denote the natural frequency of the powder oscillations.

On the curve $r = (k-1)^2/(k+1)$, damping is absent, and the frequency

$$\omega_0 = \frac{\sqrt{k(k+1)}}{(k-1)^2}. \quad (2.4)$$

The introduced quantities λ and ω are dimensionless; transition to dimensional λ and ω requires that these quantities be divided by $\kappa/(u^0)^2$, i. e., by the characteristic relaxation time of the heated layer of the condensed phase. For $\kappa = 10^3 \text{ cm}^2/\text{sec}$, we have $\kappa/(u^0)^2 = 10^{-1} \text{ sec}$ for $u^0 = 10^{-1} \text{ cm/sec}$, and 10^{-3} sec for $u^0 = 1 \text{ cm/sec}$.

The nature of the oscillatory mode depends on the relaxation between λ and ω . For $\lambda \ll \omega$, i. e., for weak damping, there is only a slight change in amplitude over one period. It is obvious that when the stability limit is approached, i. e., at $r \rightarrow (k-1)^2/(k+1)$, damping decreases, and its value may be considered arbitrarily small. At the stability limit itself, the damping constant $\lambda = 0$.

Hence, it may be assumed that in the region $k > 1$, the powder represents an oscillatory system with a definite frequency and damping constant. Powder that (at constant pressure) is brought out of the steady combustion state will relax to the steady-state regime in such a way that its burning rate performs oscillations about the steady-state value with frequency $\sqrt{\omega^2 - \lambda^2}$ and damping λ . For $k < 1$, the relaxation to the steady-state regime will be viscous in nature, i. e., without passage through the steady-state regime. Of course, the aperiodic regime can also be reached at $k > 1$ by setting $\lambda \geq \omega$; however, this condition leads to values of r so large that they are apparently never realized in practice.

At constant pressure, the oscillations will be termed free oscillations. In the following paragraph we will examine the burning of powders at harmonically varying pressure. The oscillations of the burning rate in these conditions will be termed forced oscillations.

3. Forced oscillations of the burning rate. We will examine in the linear approximation the steady-state process of powder combustion at harmonically varying pressure. As in the preceding case, the inertia properties of the heated layer of the condensed phase only will be considered. This requirement leads to an upper bound for the frequency of the pressure oscillations. Indeed, in order that the zone in which the gas is heated to the burning temperature may be considered inertialess, it is necessary that the frequency of the pressure variation be much less than the reciprocal characteristic time of the gas heating zone

$$t^* = \frac{D\rho^{*2}}{u^2\rho^2},$$

where D is diffusion coefficient, and ρ^* and ρ are the density of the gas and the solid phase, respectively. The magnitude of t^* is of the order of $\sim 10^{-5} \text{ sec}$ (see, for example, [5]). Hence, for pressure varying with frequency up to several tens of kc, the processes in the gas phase may be considered inertialess. The one inertial process takes place in the solid phase, where the following heat conduction equation holds:

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} - u \frac{\partial T}{\partial x}, \quad T|_{x=0} = T_s, \quad T|_{x=-\infty} = T_0. \quad (3.1)$$

Let us pass to the dimensionless variables

$$\theta = \frac{T - T_0}{T_s - T_0}, \quad \xi = \frac{u^0}{\kappa} x, \quad \tau = \frac{(u^0)^2}{\kappa} t, \quad v = \frac{u}{u^0} = \frac{m}{m^0}, \quad (3.2)$$

we then obtain

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \xi^2} - v \frac{\partial \theta}{\partial \xi}, \quad \theta|_{\xi=0} = \theta, \\ \theta|_{\xi=-\infty} = 0 \quad \left(\theta = \frac{T_s - T_0}{T_s - T_0} \right). \quad (3.3)$$

The steady-state regime is described by the solution

$$\theta = e^{\xi}, \quad v = 1, \quad \theta = 1, \quad \varphi = 1 \quad \varphi = (\partial \theta / \partial \xi)_0 = f / f_0, \quad (3.4)$$

where φ is a dimensional gradient.

Let us examine the fully developed combustion process at a pressure that varies according to the law

$$p = p^0 (1 + h \cos \gamma \tau),$$

where γ is dimensionless frequency. In the linear approximation, we may use the complex method. We

represent the rate, temperature, gradient, and pressure in the form

$$v = 1 + v_1 e^{i\gamma\tau}, \quad \theta = \theta^{\bar{z}} (1 + \theta_1 e^{i\gamma\tau}), \quad \vartheta = 1 + \vartheta_1 e^{i\gamma\tau},$$

$$\varphi = 1 + \varphi_1 e^{i\gamma\tau} \quad \eta = 1 + h e^{i\gamma\tau} \quad (\eta = p / p^0). \quad (3.5)$$

From relations (1.3), it is easy to obtain

$$v_1 = \frac{k}{k+r-1} \varphi_1 + \frac{v(r-1) - \mu k}{k+r-1} h,$$

$$\vartheta_1 = \frac{r}{k+r-1} \varphi_1 + \frac{\mu(k-1) - v_r}{k+r-1} h. \quad (3.6)$$

Linearization of the heat conduction equation (3.2) leads to the equation

$$i\gamma\theta_1 = \theta_1'' + \theta_1' + v_1 \quad (3.7)$$

(the prime denotes differentiation with respect to ξ). Its solution is

$$\theta_1 = A e^{z_1 \xi} + i \frac{v_1}{\gamma}, \quad (3.8)$$

$$z_1 = 1/2(\gamma/R_1 - 1) + iR_1, \quad R_1 = (1/8(\sqrt{16\gamma^2 + 1} - 1))^{1/2}. \quad (3.9)$$

The second root of the characteristic equation is neglected since it corresponds to a solution $\xi \rightarrow -\infty$. From (3.8) we obtain

$$\vartheta_1 = A + i v_1 / \gamma, \quad (3.10)$$

$$\varphi = \frac{\partial \theta}{\partial \xi} \Big|_{\xi=0} = 1 + \vartheta_1 + \frac{\partial \theta_1}{\partial \xi} \Big|_{\xi=0}, \quad \varphi_1 = \vartheta_1 + A z_1. \quad (3.11)$$

The four algebraic equations ((3.6, 7), (3.10, 11)) represent a system in closed form from which it is possible to determine φ_1 , ϑ_1 , A , and v_1 . For the most important quantity—the burning rate—we have

$$v_1 = \frac{v + (vr - \mu k) z_1}{1 - k + (r - ik/\gamma) z_1}. \quad (3.12)$$

After introduction of the quantities

$$a = v + 1/2(vr - \mu k)(\gamma/R_1 - 1), \quad b = (vr - \mu k) R_1,$$

$$c = 1 + \left(\frac{\gamma}{R_1} - 1\right) \left(\frac{r}{2} - \frac{kR_1}{\gamma}\right), \quad d = rR_1 - \frac{k}{2\gamma} \left(\frac{\gamma}{R_1} - 1\right), \quad (3.13)$$

the complex amplitude of the burning rate, its modulus, and phase shift with respect to pressure can be written in the form

$$v_1 = \frac{a + ib}{c + id}, \quad |v_1| = \left(\frac{a^2 + b^2}{c^2 + d^2}\right)^{1/2}, \quad \text{tg } \psi = \frac{bc - ad}{ac + bd}. \quad (3.14)$$

For constant temperature, these expressions take the form

$$|v_1| = \frac{v}{\sqrt{c_0^2 + d_0^2}}, \quad \text{tg } \psi = -\frac{d_0}{c_0},$$

$$\left(c_0 = 1 - k \left(1 - \frac{R_1}{\gamma}\right), \quad d_0 = -\frac{k}{2R_1} \left(1 - \frac{R_1}{\gamma}\right)\right).$$

The results are the same as those obtained by Zel'dovich [3].

4. Resonance. Let us examine the obtained relations in the case where the frequency of the pressure oscillations approaches the natural frequency of the pressure oscillations approaches the natural frequency of the oscillations, i. e., the case of resonance. Let $\gamma = \omega + \epsilon$. We pass from the quantities k , r , and γ to k , λ , and ϵ , assuming that λ and ϵ are small compared to the natural frequency ω . In the approximation employed

$$r = \frac{(k-1)^2}{k+1} \left[1 + 2\left(\frac{k-1}{k+1}\right)^2 \lambda\right],$$

$$\gamma = \frac{\sqrt{k}(k+1)}{(k-1)^2} \left[1 + \frac{(k-1)^2}{\sqrt{k}(k+1)} \epsilon - 2\left(\frac{k-1}{k+1}\right)^2 \lambda\right],$$

$$R_1 = \frac{\sqrt{k}}{|k-1|} \left[1 + \frac{(k-1)^2(k+1)}{\sqrt{k}(k^2+6k+4)} \epsilon - \frac{2(k-1)^2}{k^2+6k+4} \lambda\right]. \quad (4.1)$$

Substitution of these expressions into (3.12) yields (for $k > 1$)

$$v_1 = \quad (4.2)$$

$$= \frac{(k+1)(k^2+6k+1)\{v(k^2-1) + [v(k-1)^2 - \mu k(k+1)](1+i\sqrt{k})\}}{2(k-1)^4[-\sqrt{k}(k-1) + i(3k+1)](\epsilon - i\lambda)} h.$$

For the modulus of the amplitude we obtain

$$|v_1|^2 = \quad (4.3)$$

$$= \frac{k(k+1)^2(k^2+6k+1)[v^2(k-1)^2 - 2\mu vk(k-1) + \mu^2 k(k+1)]}{4(k-1)^8(\epsilon^2 + \lambda^2)} h^2,$$

i. e., a typical resonance dependence of the burning rate on frequency. If $k < 1$, resonance effects do not arise. In physical terms, this is explained by the fact that powder has no natural frequency, while formally, it is expressed by a specific dependence of R_1 on the quantity $k - 1$. For the phase shift we have

$$\text{tg } \psi = \frac{F\lambda - G\epsilon}{G\lambda + F\epsilon},$$

$$F = v(k-1)^2 - 2\mu k(k+1),$$

$$G = \sqrt{k}[v(k-1)(k+3) + \mu(k+1)^2]. \quad (4.4)$$

As is usually the case, on passage through the resonance point (variation of ϵ from $-\lambda$ to λ), the phase changes by $1/2 \pi$. Note that in the general case both the equation and boundary conditions are nonlinear. It is of interest to study the nonlinear oscillations of the burning rate which arise at large pressure amplitudes. In this case, one may expect all the characteristic phenomena associated with nonlinear oscillations, namely, a nonunique dependence of the burning rate on the frequency of the pressure variations, abrupt changes from one mode of burning to another, resonances at frequencies that do not match the

natural frequency of the system, and so forth (see, for example, [6]).

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